

COLORED GRAPHS, GAUSSIAN INTEGRALS AND STABLE GRAPH POLYNOMIALS.

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Asymptotic expansions of Gaussian integrals may often be interpreted as generating functions for certain combinatorial objects (graphs with additional data). In this article we discuss a general approach to all such cases using colored graphs. We prove that the generating power series for such graphs satisfy the same system of partial differential equations as the Gaussian integral and the formal power series solution of this system is unique. The solution is obtained as the genus expansion of the generating power series. The initial term of this expansion is the corresponding generating function for trees. The consequence equations for this term turns to be equivalent to the inversion problem for the gradient mapping defined by the initial condition. The equations for the higher terms of the genus expansion are linear. The solutions of these equations can be expressed explicitly by substitution of the initial conditions and the initial term (the tree expansion) into some universal polynomials (for $g > 1$) which are generating functions for stable closed graphs. (For $g = 1$ instead of polynomials appears logarithm.) The stable graph polynomials satisfy certain recurrence. In [1] some of these results were obtained for $r = 1$ by more or less direct solution of differential equations. Here we present purely combinatorial proofs.

1. INTRODUCTION.

First let us fix the notations. Consider vectors $X = (x_1, \dots, x_r)$ and a symmetric $r \times r$ matrix $S = (s_{ij})$. We shall consider x_i and s_{ij} and \hbar as independent commutative variables.

For a multi-index $N = (n_1, \dots, n_r)$ we shall use the notations $X^N = x_1^{n_1} \cdot \dots \cdot x_r^{n_r}$, $N! = n_1! \dots n_r!$ and $|N| = \sum n_i$. $N \geq 0$ will mean that all $n_i \geq 0$. For the multi-index $(0, \dots, 0, 1, 0, \dots, 0)$ (all zeros except 1

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in the position i) we shall use the notation $\{i\}$:

$$(1.1) \quad \{i\} = (0, \dots, 0, 1, 0, \dots, 0).$$

Next we denote $\{ij\} = \{i\} + \{j\}$, so that for $i \neq j$ the multi-index $\{ij\}$ has exactly two non-zero positions i and j and for $i = j$ the multi-index $\{ii\}$ has 2 in the non-zero positions i . Thus for $i \neq j$ $\{ij\}! = 1$ and $\{ii\}! = 2$. As well we may use multi-indices $\{ijk\} = \{i\} + \{j\} + \{k\}$ and so on.

We start with a formal power series $U(X, \hbar) \in \frac{1}{\hbar} \mathbb{C}[[X, \hbar]]$ which we shall write as a Taylor expansion

$$(1.2) \quad U(X, \hbar) = \sum_{g \geq 0} \sum_{N \geq 0} a_{g,N} \frac{X^N}{N!} \hbar^{g-1}.$$

It is convenient to consider the coefficients $a_{g,N}$ as independent variables.

A formal definition of modular graph see in [1]. Informally speaking modular graph is a graph which may have edges with only one end. Such edges are usually called *tails*. Now we want to label vertices of a modular graph by the variables $a_{g,N}$, tails will be labeled by x_1, \dots, x_r and the edges will be labeled by the elements of the symmetric matrix S . For that purpose we need to mark the half-edges incident to each vertex by the numbers $1, \dots, r$ so that two half-edges of one edge may be marked by different indices. Thus it is natural to insert a new two-valent vertex into the middle of the edge which will break the edge into two new edges. Now each of them can be marked by one index $i \in \{1, \dots, r\}$. This leads to the following formal definition.

Definition 1.1. *A bipartite colored modular graph is a collection of the following data:*

- (1) *a modular graph Γ whose set of vertices $V(\Gamma)$ is a disjoint union of two partite sets $V(\Gamma) = V_a(\Gamma) \sqcup V_s(\Gamma)$;*
- (2) *a mapping from the set of edges and tails of Γ to the set $\{1, \dots, r\}$ which we shall call coloring;*
- (3) *a mapping $g : V_a(\Gamma) \rightarrow \{0, 1, 2, \dots\}$; nonnegative integer $g(v)$ will be called genus of the vertex g . We shall call a graph combinatorial if for all its vertices $g(v) = 0$.*

These data should satisfy the following properties:

- (1) *two vertices from the same partite sets are not connected by an edge;*
- (2) *vertices from $V_s(\Gamma)$ should be only two-valent and should have no incident tails.*

The vertices from $V_s(\Gamma)$ will be called *s-vertices* and the vertices from $V_a(\Gamma)$ will be called *a-vertices*.

Consider a bipartite colored modular graph Γ . *Genus* of Γ is defined by

$$(1.3) \quad g(\Gamma) = \sum_{v \in V_a(\Gamma)} g(v) + b_1(\Gamma) - b_0(\Gamma) + 1,$$

where $b_m(\Gamma)$ is the m -th Betti number of the graph (considered as a 1-dimensional simplicial complex). Thus for a connected graph Γ

$$(1.4) \quad g(\Gamma) = \sum_{v \in V_a(\Gamma)} g(v) + b_1(\Gamma).$$

For a graph Γ let us fix a multi-index $N(\Gamma) = (n_1, \dots, n_r)$ where n_i are the numbers of and tails colored by i . A graph without tails (i.e. $N(\Gamma) = (0, \dots, 0)$) will be called *closed*. *Valence* of an *a*-vertex v of a colored graph Γ is a multi-index $N(v) = (\nu_1(v), \dots, \nu_r(v))$, where $\nu_i(v)$ is the number of edges and tails colored by i and incident to v .

Now to each *a*-vertex $v \in V_a(\Gamma)$ we attach the variable $a_{g(v), N(v)}$ and we attach s_{ij} to an *s*-vertex $v \in V_s(\Gamma)$ if v is incident to two edges colored by i and j . Thus for each colored graph Γ we define the monomial

$$(1.5) \quad \mu(\Gamma) = \prod_{v \in V_a(\Gamma)} a_{g(v), N(v)} \prod_{v \in V_a(\Gamma)} s_{ij}.$$

For $N = (n_1, \dots, n_r)$, $g \geq 0$ denote by $\tilde{\mathcal{B}}_{g,N}$ the set of all colored bipartite graphs Γ having n_i half-edges of the color i and $g(\Gamma) = g$ and by $\mathcal{B}_{g,N}$ the set of all such connected bipartite graphs. Put $\tilde{\mathcal{B}}_g = \bigcup_{N \geq 0} \tilde{\mathcal{B}}_{g,N}$ and $\mathcal{B}_g = \bigcup_{N \geq 0} \mathcal{B}_{g,N}$; $\tilde{\mathcal{B}} = \bigcup_{g \geq 0} \tilde{\mathcal{B}}_g$ and $\mathcal{B} = \bigcup_{g \geq 0} \mathcal{B}_g$.

Define two generating series

$$(1.6) \quad \Psi(S, X, \hbar) = \sum_{\Gamma \in \mathcal{B}} \frac{1}{|\text{Aut } \Gamma|} \mu(\Gamma) X^{N(\Gamma)} \hbar^{g(\Gamma)-1} \in \frac{1}{\hbar} \mathbb{C}[[X, S, \hbar]];$$

$$(1.7) \quad \tilde{\Psi}(S, X, \hbar) = \sum_{\Gamma \in \tilde{\mathcal{B}}} \frac{1}{|\text{Aut } \Gamma|} \mu(\Gamma) X^{N(\Gamma)} \hbar^{g(\Gamma)-1} \in \frac{1}{\hbar} \mathbb{C}[[X, S, \hbar, \frac{1}{\hbar}]].$$

A standard combinatorial principle says that the generating function for all graphs is the exponent of the generating function for connected graphs.

Theorem 1.1.

$$(1.8) \quad \tilde{\Psi}(X, S, \hbar) = \exp [\Psi(S, X, \hbar)].$$

Note that substituting $S = 0$ into (1.6) we get the generating power series for distinct a -vertices i.e.

$$(1.9) \quad \Psi(X, 0, \hbar) = \sum_{g \geq 0} \sum_{N \geq 0} a_{g,N} \frac{X^N}{N!} \hbar^{g-1} = U(X, \hbar).$$

We prove that the generating series $\tilde{\Psi}$ is the unique solution of the system of linear partial differential equations generalizing the heat equation and the series Ψ is the unique solution of the system of non-linear partial differential equations generalizing the Burgers equation (see [1]).

Theorem 1.2. 1) *The generating series (1.7) for all colored modular graphs*

$$\tilde{\Psi}(S, X, \hbar) = \sum_{\Gamma \in \tilde{\mathcal{B}}} \frac{1}{|\text{Aut } \Gamma|} \mu(\Gamma) X^{N(\Gamma)} \hbar^{g(\Gamma)-1} \in \frac{1}{\hbar} \mathbb{C}[[X, S, \hbar, \frac{1}{\hbar}]]$$

is the unique solution in $\mathbb{C}[[x, S, \hbar, \frac{1}{\hbar}]]$ of the equations

$$(1.10) \quad \frac{\partial \tilde{\Psi}}{\partial s_{ij}} = \frac{\hbar}{\{ij\}!} \frac{\partial^2 \tilde{\Psi}}{\partial x_i \partial x_j}$$

with the initial condition

$$(1.11) \quad \tilde{\Psi}(X, 0, \hbar) = \exp[U(X, \hbar)]$$

and $\tilde{\Psi}$ provides the formal asymptotic expansion of the Gaussian integral:

$$(1.12) \quad \tilde{\Psi}(x, S, \hbar) \sim \frac{1}{(2\pi\hbar)^{r/2} (\det S)^{1/2}} \int \exp \left[U(\xi, \hbar) - \frac{1}{2\hbar} (X - \xi)^T S^{-1} (X - \xi) \right] d\xi$$

2) *The generating series (1.6) for all colored modular graphs*

$$\Psi(S, X, \hbar) = \sum_{\Gamma \in \mathcal{B}} \frac{1}{|\text{Aut } \Gamma|} \mu(\Gamma) X^{N(\Gamma)} \hbar^{g(\Gamma)-1} \in \frac{1}{\hbar} \mathbb{C}[[X, S, \hbar]]$$

is the unique solution in $\frac{1}{\hbar} \mathbb{C}[[x, S, \hbar]]$ of the equations

$$(1.13) \quad \frac{\partial \Psi}{\partial s_{ij}} = \frac{\hbar}{\{ij\}!} \left[\frac{\partial^2 \Psi}{\partial x_i \partial x_j} + \left(\frac{\partial \Psi}{\partial x_i} \right) \left(\frac{\partial \Psi}{\partial x_j} \right) \right]$$

with the initial condition

$$(1.14) \quad \Psi(X, 0, \hbar) = U(X, \hbar)$$

and Ψ provides the formal asymptotic expansion of the Gaussian integral

$$(1.15) \quad \Psi(X, S, \hbar) \sim \log \frac{1}{(2\pi\hbar)^{r/2}(\det S)^{1/2}} \int \exp \left[U(\xi, \hbar) - \frac{1}{2\hbar}(X - \xi)^T S^{-1}(X - \xi) \right] d\xi$$

Note that the statements concerning Gaussian integrals are more or less trivial. For a positive matrix S the integral (1.12) is simply the average of the function U by the normal distribution having mean value X and covariance matrix $\hbar S$. It is easy to verify that this integral considered as a function on X and S satisfies the system (1.10). Thus its asymptotic expansion should also satisfy it. Now it is clear that the formal power series solution of (1.10) and (1.13) with the corresponding initial condition exist and is unique.

Next let us consider the genus expansion of Ψ

$$(1.16) \quad \Psi(S, X, \hbar) = \sum_{g \geq 0} \Psi_g(S, X) \hbar^{g-1},$$

where $\Psi_g(S, X) \in \mathbb{C}[[X, S]]$.

For $g = 0$ (1.13) provides the equation

$$(1.17) \quad \frac{\partial \Psi_0}{\partial s_{ij}} = \frac{1}{\{ij\}!} \left(\frac{\partial \Psi_0}{\partial x_i} \right) \left(\frac{\partial \Psi_0}{\partial x_j} \right).$$

For $g > 0$ (1.13) provides recursive equations

$$(1.18) \quad \frac{\partial \Psi_g}{\partial s_{ij}} = \frac{1}{\{ij\}!} \left[\frac{\partial^2 \Psi_{g-1}}{\partial x_i \partial x_j} + \sum_{m=0}^g \left(\frac{\partial \Psi_m}{\partial x_i} \right) \left(\frac{\partial \Psi_{g-m}}{\partial x_j} \right) \right].$$

The equation (1.17) looks better for the gradient vector function

$$(1.19) \quad \Phi(S, X) = \nabla_X \Psi_0(S, X) = (\Phi_1(S, X), \dots, \Phi_r(S, X)).$$

where $\Phi_i(S, X) = \frac{\partial \Psi_0(S, X)}{\partial x_i}$. Then (1.17) provides the equations:

$$(1.20) \quad \frac{\partial \Phi_m}{\partial s_{ij}} = \Phi_i \frac{\partial \Phi_m}{\partial x_j} + \Phi_j \frac{\partial \Phi_m}{\partial x_i} \quad \text{for } i \neq j;$$

$$(1.21) \quad \frac{\partial \Phi_m}{\partial s_{ii}} = \Phi_i \frac{\partial \Phi_m}{\partial x_i} \quad \text{for } i = j.$$

Initial conditions for these equations are given by the genus expansion of the initial condition (1.14):

$$(1.22) \quad U(X, \hbar) = \sum_{g \geq 0} U_g(X) \hbar^{g-1}.$$

We shall also use the gradient vector function

$$(1.23) \quad F(X) = \nabla_X U_0(X) = (F_1(X), \dots, F_r(X)),$$

where $F_i = \frac{\partial U_0}{\partial x_i}$ and the Hessian matrix function

$$(1.24) \quad H(X) = \nabla_X F(X) = \left(\frac{\partial^2 U_0}{\partial x_i \partial x_j} \right).$$

The equations (1.20)–(1.21) may be solved explicitly. They provide the following functional equation.

Theorem 1.3. *Let $F(X) = (F_1(X), \dots, F_r(X))$ be any formal series vector. Then the solution of the system (1.20) and (1.21) with the initial condition $\Phi(0, X) = F(X)$ satisfies the functional equation*

$$(1.25) \quad \Phi(S, X) = F(X + S\Phi(S, X)).$$

It is not hard to verify this by direct calculations but we give in section 3 a combinatorial proof for this theorem.

The functional equation (3.1) is equivalent to the inversion problem for the formal mapping $A(X) = X - SF(X)$. This well-known fact was discussed for the case of diagonal matrix S in [2]. Here we shall only present the statement; the proof is quite the same as in [2].

Corollary 1.1. *Let $\Phi(S, X)$ be the solution of the system (1.20) and (1.21) with the initial condition $\Phi(0, X) = F(X)$. Consider the following formal mappings from \mathbb{C}^r to \mathbb{C}^r :*

$$(1.26) \quad A(X) = X - SF(X)$$

$$(1.27) \quad B(X) = X + S\Phi(S, X)$$

Then these mappings are inverse to each other:

$$(1.28) \quad A(B(X)) = X \text{ and } B(A(X)) = X.$$

Thus the formal Cauchy problems for the systems of the Burgers equations for Φ is equivalent to the problem of finding the inverse function for the initial conditions. Integrating Φ we may get the first term of the expansion (1.16). It is remarkable that the second term of these expansions may be presented explicitly in terms of Φ and H .

The equation (1.18) provide the following equations for Ψ_1 :

$$(1.29) \quad \frac{\partial \Psi_1}{\partial s_{ij}} = \frac{1}{\{ij\}!} \left[\frac{\partial^2 \Psi_0}{\partial x_i \partial x_j} + \Phi_i \frac{\partial \Psi_1}{\partial x_j} + \Phi_j \frac{\partial \Psi_1}{\partial x_i} \right].$$

Theorem 1.4. *The solution of the equations (1.29) or with the initial conditions $\Psi_1(0, X) = U_1(X)$ is given by:*

(1.30)

$$\Psi_1(S, X) = U_1(X + S\Phi(S, X)) - \frac{1}{2} \operatorname{tr} \ln(E - SH(X + S\Phi(S, X)))$$

Probably it is still possible to prove this theorem by direct calculation but we present a combinatorial proof of it in section 3.

For $g > 1$ the recurrent equations (1.18) for Ψ_g are linear on Ψ_g . The solution may be expressed in terms of Φ , H and the *stable graph polynomials*. A graph is called *stable* if all its genus zero vertices are at least trivalent and all its genus one vertices are at least univalent. In section 3 we introduce for $g > 1$ *stable graph polynomials* $P_g(\{a_{g,N}\}, S)$ (see (3.38)) depending on independent variables $a_{g,N}$ for all $g \geq 0$, $|N| \geq 3$ and symmetric matrix S as generating functions for stable graphs. The stable graph polynomials satisfy certain recurrence (see theorem 4.1) and certain homogeneity properties (see theorem 3.5). The solution of (1.18) is expressed by the stable graph polynomials as follows (see theorem 3.1):

$$(1.31) \quad \Psi_g(S, X) = P_g \left(\left\{ a_{g,N} := \frac{\partial^{|N|} U_g(X + S\Phi(S, X))}{\partial X^N} \right\}, (E - SH(X + S\Phi(S, X)))^{-1} S \right);$$

We may arrange the stable graph polynomials into the generating power series

$$(1.32) \quad \mathcal{P}(\{a_{g,N}\}, S, \hbar) = \sum_{g \geq 2} P_g(\{a_{g,N}\}, S) \hbar^{g-1}.$$

Substituting $X = 0$ into (1.15) we obtain another useful asymptotic expansion.

Theorem 1.5. *If $a_{g,N} = 0$ for $|N| + 2g - 2 \leq 0$ then the series (1.32) provides the asymptotic expansion*

(1.33)

$$\mathcal{P}(\{a_{g,N}\}, S, \hbar) \sim \log \frac{1}{(2\pi\hbar)^{r/2} (\det S)^{1/2}} \int \exp \left[U(\xi, \hbar) - \frac{1}{2\hbar} \xi^T S^{-1} \xi \right] d\xi,$$

where $U(X, \hbar) = \sum_{g \geq 0} \sum_{N \geq 0} a_{g,N} \frac{X^N}{N!} \hbar^{g-1}$.

Note that the case $r = 1$ which is far from being trivial. For this case we have one variable s corresponding to edges of a graph and two-index variables $a_{g,n}$. Denoting by \mathcal{A}_g^k the set of genus g stable closed graphs

we define

$$(1.34) \quad P_g(\{a_{m,N}\}, s) = \sum_{k=0}^{3g-3} \sum_{\Gamma \in \mathcal{A}_g^k} \frac{\mu(\Gamma)}{|\text{Aut } \Gamma|} s^k.$$

(Stable genus g graph without half-edges has at most $3g - 3$ edges.)
For instance for $g = 2$

$$(1.35) \quad P_2 = a_{2,0} + \frac{1}{2}a_{1,1}^2 s + \frac{1}{2}a_{1,2}s + \frac{1}{2}a_{1,1}a_{0,3}s^2 + \frac{1}{8}a_{0,4}s^2 + \frac{5}{24}a_{0,3}^2 s^3.$$

There are two interesting specializations of the variables $\{a_{g,n}\}$: counting functions for all combinatorial graphs and counting functions for all stable combinatorial graphs and

For the counting functions for all combinatorial graphs we put

$$(1.36) \quad a_{g,n}^{\text{comb}} = \begin{cases} 1 & \text{if } g = 0 \\ 0 & \text{otherwise,} \end{cases}$$

and for the counting functions for all stable combinatorial graphs we put

$$(1.37) \quad a_{g,n}^{\text{st}} = \begin{cases} 1 & \text{if } g = 0 \quad \text{and } n \geq 3 \\ 0 & \text{otherwise.} \end{cases}$$

The function Φ satisfies the functional equation

$$(1.38) \quad \Phi^{\text{comb}}(s, x) = e^{x+s\Phi^{\text{comb}}(s,x)}$$

for the counting functions for all combinatorial graphs¹ and the functional equation

$$(1.39) \quad e^{x+s\Phi^{\text{st}}(s,x)} = 1 + x + (s+1)\Phi^{\text{st}}(s, x)$$

for the counting functions for all stable combinatorial graphs.

Stable graph polynomials for both cases coincide; we denote

$$(1.40) \quad P_g^{\text{comb}}(s) = P_g(\{a_{m,N} := a_{m,N}^{\text{comb}}\}) = P_g(\{a_{m,N} := a_{m,N}^{\text{st}}\}, s).$$

For instance for $g = 2$

$$(1.41) \quad P_2^{\text{comb}} = \frac{1}{8}s^2 + \frac{5}{24}s^3.$$

In section 5 we prove the following theorem.

¹Note that $\Phi^{\text{comb}}(s, 0)$ is the classical generating function for rooted trees (without half-edges) whose coefficients are given by the well-known Cayley formula:

$$\Phi^{\text{comb}}(s, 0) = \sum_{k=0}^{\infty} \frac{(k+1)^k}{(k+1)!} s^k.$$

Theorem 1.6. 1) Combinatorial stable graph polynomials P_g^{comb} for $g > 2$ satisfy the recurrence

$$(1.42) \quad \frac{dP_g^{\text{comb}}}{ds} = \frac{1}{2} \left[D_{\text{comb}}^2(P_{g-1}^{\text{comb}}) + 2sD_{\text{comb}}(P_{g-1}^{\text{comb}}) + \sum_{m=2}^{g-2} D_{\text{comb}}(P_m^{\text{comb}})D_{\text{comb}}(P_{g-m}^{\text{comb}}) \right],$$

where

$$(1.43) \quad D_{\text{comb}} = s(s+1)\frac{d}{ds} - (g-1).$$

2) For $g \geq 2$ the counting function for all combinatorial graphs

$$(1.44) \quad \Psi_g^{\text{comb}}(s, x) = \frac{1}{\Phi^{\text{comb}}(s, x)^{g-1}} P_g^{\text{comb}} \left(\frac{s\Phi^{\text{comb}}(s, x)}{1 - s\Phi^{\text{comb}}(s, x)} \right).$$

3) For $g \geq 2$ the counting function for all stable combinatorial graphs

$$(1.45) \quad \Psi_g^{\text{st}}(s, x) = \frac{1}{(1+x+(s+1)\Phi^{\text{st}}(s, x))^{g-1}} P_g^{\text{comb}} \left(\frac{s(1+x+(s+1)\Phi^{\text{st}}(s, x))}{1 - s(x+(s+1)\Phi^{\text{st}}(s, x))} \right).$$

Formula (1.45) was derived in [1] by direct solution of the equations (1.18).

2. BIPARTITE COLORED GRAPHS.

First let us prove that $\tilde{\Psi}(X, S, \hbar)$ satisfies the system (1.10). For this purpose we need to interpret the derivatives of $\tilde{\Psi}(X, S, \hbar)$ and $\tilde{\Psi}(X, S, \hbar)$ as certain generating power series.

Define the sets of bipartite colored modular (connected) graphs with one marked ij -valent s -vertex by $\tilde{\mathcal{B}}_{g,N}^{[ij]}$ ($\mathcal{B}_{g,N}^{[ij]}$). For a graph $\Gamma \in \tilde{\mathcal{B}}_{g,N}^{[ij]}$ (or $\mathcal{B}_{g,N}^{[ij]}$) definition of $\mu(\Gamma)$ should be improved: we put

$$(2.1) \quad \mu^{[ij]}(\Gamma) = \prod_{\substack{\text{nonmarked} \\ v \in V_s(\Gamma)}} s_{ij} \prod_{v \in V_a(\Gamma)} a_{g(v), \nu(v)}.$$

Fix $L = (l_1, \dots, l_r)$, $N = (n_1, \dots, n_r)$. The set of bipartite colored modular (connected) graphs with $n_i + l_i$ tails of the color i , l_i of them marked and ordered, will be denoted by $\tilde{\mathcal{B}}_{g,N,[L]}$ ($\mathcal{B}_{g,N,[L]}$).

It is clear that

$$(2.2) \quad \sum_{-\infty < g < +\infty} \sum_{N \geq 0} \left(\sum_{\Gamma \in \tilde{\mathcal{B}}_{g,N}^{[ij]}} \frac{\mu(\Gamma)}{|\text{Aut } \Gamma|} \right) X^N \hbar^{g-1} = \frac{\partial \tilde{\Psi}(S, X, \hbar)}{\partial s_{ij}},$$

$$(2.3) \quad \sum_{-\infty < g < +\infty} \sum_{N \geq 0} \left(\sum_{\Gamma \in \tilde{\mathcal{B}}_{g,N,[L]}^K} \frac{\mu(\Gamma)}{|\text{Aut } \Gamma|} \right) X^N \hbar^{g-1} = \frac{\partial^l \tilde{\Psi}(S, X, \hbar)}{\partial x_1^{l_1} \dots \partial x_r^{l_r}},$$

and the same is true for the generating series for connected graphs (i.e. for Ψ without tilde):

$$(2.4) \quad \sum_{g \geq 0} \sum_{N \geq 0} \left(\sum_{\Gamma \in \mathcal{B}_{g,N}^{[ij]}} \frac{\mu(\Gamma)}{|\text{Aut } \Gamma|} \right) X^N \hbar^{g-1} = \frac{\partial \Psi(S, X, \hbar)}{\partial s_{ij}},$$

$$(2.5) \quad \sum_{g \geq 0} \sum_{N \geq 0} \left(\sum_{\Gamma \in \mathcal{B}_{g,N,[L]}} \frac{\mu(\Gamma)}{|\text{Aut } \Gamma|} \right) X^N \hbar^{g-1} = \frac{\partial^l \Psi(S, X, \hbar)}{\partial x_1^{l_1} \dots \partial x_r^{l_r}},$$

There is a natural clutching map $\tilde{\mathcal{B}}_{g,N,[\{ij\}]} \rightarrow \tilde{\mathcal{B}}_{g+1,N}^{[ij]}$: to clutch together two tails we insert a ij -valent s -vertex between them. This map is bijective for $i \neq j$ and is 2-fold covering for $i = j$ (since there are two possible orderings on the set of two marked tails). Hence the generating power series coincide up to factor $\frac{\hbar}{\{ij\}!}$. So we get exactly the equation (1.10). Thus we have proved the theorem 1.2.

3. GENUS EXPANSION.

Next let us consider the genus expansion of Ψ (1.16):

$$\Psi(S, X, \hbar) = \sum_{g \geq 0} \Psi_g(S, X) \hbar^{g-1},$$

As we have seen in the Introduction for $g = 0$ the equations (1.13) provide the equations (1.17)

$$\frac{\partial \Psi_0}{\partial s_{ij}} = \frac{1}{\{ij\}!} \left(\frac{\partial \Psi_0}{\partial x_i} \right) \left(\frac{\partial \Psi_0}{\partial x_j} \right).$$

For $g > 0$ (1.13) provides recursive equations (1.18)

$$\frac{\partial \Psi_g}{\partial s_{ij}} = \frac{1}{\{ij\}!} \left[\frac{\partial^2 \Psi_{g-1}}{\partial x_i \partial x_j} + \sum_{m=0}^g \left(\frac{\partial \Psi_m}{\partial x_i} \right) \left(\frac{\partial \Psi_{g-m}}{\partial x_j} \right) \right].$$

For the gradient vector functions

$$\Phi(S, X) = \nabla_X \Psi_0(S, X) = (\Phi_1(S, X), \dots, \Phi_r(S, X)),$$

$$F(X) = \nabla_X U_0(X) = (F_1(X), \dots, F_r(X)),$$

and the Hessian matrix function

$$H(X) = \nabla_X F(X) = \left(\frac{\partial^2 U_0}{\partial x_i \partial x_j} \right).$$

we got the equations (1.20) and (1.21)

$$\frac{\partial \Phi_m}{\partial s_{ij}} = \Phi_i \frac{\partial \Phi_m}{\partial x_j} + \Phi_j \frac{\partial \Phi_m}{\partial x_i} \quad \text{for } i \neq j;$$

$$\frac{\partial \Phi_m}{\partial s_{ii}} = \Phi_i \frac{\partial \Phi_m}{\partial x_i} \quad \text{for } i = j.$$

Let us prove the theorem 1.3:

Theorem. 1.3 *Let $F(X) = (F_1(X), \dots, F_r(X))$ be any formal series vector. Then the solution of the system (1.20) and (1.21) with the initial condition $\Phi(0, X) = F(X)$ satisfies the functional equation*

$$(3.1) \quad \Phi(S, X) = F(X + S\Phi(S, X)).$$

Note that according to (2.5) the series $\Phi_i(S, X)$ is the generating function for the genus 0 connected trees from $\mathcal{B}_{0, \{i\}} = \bigcup_{N \geq 0} \mathcal{B}_{0, N, \{i\}}$ (trees with one marked tail of the color i). Consider the set $\mathcal{B}_{0, \{i\}}^0$ of edgeless genus 0 connected trees with one marked half-edge of the color i $\mathcal{B}_{0, \{i\}}^0$ (i. e. $\mathcal{B}_{0, \{i\}}^0$ is the set of single vertices). Attaching to each graph $\Gamma \in \mathcal{B}_{0, \{i\}}$ the vertex adjacent to the marked tail provides the mapping

$$(3.2) \quad c_0: \mathcal{B}_{0, \{i\}} \rightarrow \mathcal{B}_{0, \{i\}}^0.$$

For any $\Delta \in \mathcal{B}_{0, \{i\}}^{(0)}$ (Δ consists of a single vertex with several tails and one marked tail of the color i) all the graphs in $c_1^{-1}(\Delta)$ are constructed from Δ by clutching arbitrary genus 0 trees with one marked tail to some of the tails of Δ (inserting a two-valent s -vertex between the corresponding tail of Γ and the marked edge of the tree). Therefore

$$(3.3) \quad \sum_{\Gamma \in c_0^{-1}(\Delta)} \frac{\mu(\Gamma)}{|\text{Aut } \Gamma|} X^{N(\Gamma)} = \frac{\mu(\Delta)}{|\text{Aut } \Delta|} (X + S\Phi(S, X))^{N(\Delta)}.$$

But the generating function for $\mathcal{B}_{0, \{\{i\}\}}^{(0)}$ is $\Phi_i(0, X) = F_i(X)$ and therefore taking the sum over all $\Delta \in \mathcal{B}_{0, \{\{i\}\}}^{(0)}$ we get

$$\begin{aligned}
 (3.4) \quad \Phi_i(S, X) &= \sum_{\Gamma \in \mathcal{B}_{0, \{\{i\}\}}} \frac{\mu(\Gamma)}{|\text{Aut } \Gamma|} X^{N(\Gamma)} = \sum_{\Delta \in \mathcal{B}_{0, \{\{i\}\}}^{(0)}} \sum_{\Gamma \in c_0^{-1}(\Delta)} \frac{\mu(\Gamma)}{|\text{Aut } \Gamma|} X^{N(\Gamma)} = \\
 &= \sum_{\Delta \in \mathcal{B}_{0, \{\{i\}\}}^{(0)}} \frac{\mu(\Delta)}{|\text{Aut } \Delta|} (X + S\Phi(S, X))^{N(\Delta)} = F_i(X + S\Phi(S, X)).
 \end{aligned}$$

The functional equation (3.1) is equivalent to the inversion problem for the formal mapping $A(X) = X - SF(X)$.

Corollary 3.1. *Let $\Phi(S, X)$ be the solution of the system (1.20) and (1.21) with the initial condition $\Phi(0, X) = F(X)$. Consider the following formal mappings from \mathbb{C}^r to \mathbb{C}^r :*

$$(3.5) \quad A(X) = X - SF(X)$$

$$(3.6) \quad B(X) = X + S\Phi(S, X)$$

Then these mappings are inverse to each other:

$$(3.7) \quad A(B(X)) = X \text{ and } B(A(X)) = X.$$

Differentials of inverse mappings are also inverse to each other. Consider the Hessian matrix

$$(3.8) \quad \Theta(S, X) = \nabla_X \Phi(S, X) = \left(\frac{\partial^2 \Psi_0}{\partial x_i \partial x_j} \right).$$

Note that $\Theta(0, X)$ is the Hessian matrix of the initial condition (1.24): $\Theta(0, X) = H(X)$. Thus we get the following equation for Θ .

Corollary 3.2.

$$\begin{aligned}
 (3.9) \quad E + S\Theta(S, X) &= (E - S\Theta(0, X + S\Phi(S, X)))^{-1} = \\
 &= (E - SH(X + S\Phi(S, X)))^{-1}
 \end{aligned}$$

The following considerations will provide us an independent combinatorial proof of corollary 3.9.

An s -vertex $v' \in V_s(\Gamma)$ of a colored bipartite connected modular graph $\Gamma \in \mathcal{B}_{g, N}$ will be called *1-cut* if deletion of v' disconnects the graph and at least one of the two new connected components has genus zero. A graph without 1-cuts will be called *2-connected*. Pick a graph $\Gamma \in \mathcal{B}_{g, N}$. Let us mark all the vertices $v'' \in V_a(\Gamma)$ which are connected by an edge with at least one vertex $v' \in V_s(\Gamma)$ which is not a 1-cut

and all the the vertices $v'' \in V_a(\Gamma)$ with $g(v'') > 0$. (Note that a genus 0 graph will have no marked vertices.) Next let us delete all the 0-cuts connected by an edge with at least one marked vertex. As the result we shall obtain a number of genus 0 connected component Γ_i , $i > 0$ and one genus g connected component Γ_1 without 1-cuts (the one having marked vertices). Let us for $g > 0$ denote the set of all colored bipartite connected and 2-connected (i.e. without 1-cuts) modular graphs $\Gamma \in \mathcal{B}_{g,N}$ by $\mathcal{B}_{g,N}^{(1)}$. The above construction provides the mapping

$$(3.10) \quad c_1 : \mathcal{B}_{g,N} \rightarrow \mathcal{B}_{g,N}^{(1)}.$$

Consider for $g > 0$ the corresponding generating function

$$(3.11) \quad \Psi_g^{(1)}(S, X) = \sum_{N \geq 0} \left(\sum_{\Gamma \in \mathcal{B}_{g,N}^{(1)}} \frac{\mu(\Gamma)}{|\text{Aut } \Gamma|} \right) X^N$$

For any $\Gamma \in \mathcal{B}_{g,N}^{(1)}$ all the graphs in $c_1^{-1}(\Gamma)$ may be constructed from Γ by clutching arbitrary genus 0 trees with one marked tail to some of the tails of Γ (inserting a two-valent s -vertex between the corresponding tail of Γ and the marked edge of the tree). Therefore

$$(3.12) \quad \sum_{\Delta \in c_1^{-1}(\Gamma)} \frac{\mu(\Delta)}{|\text{Aut } \Delta|} X^{N(\Delta)} = \frac{\mu(\Gamma)}{|\text{Aut } \Gamma|} (X + S\Phi(S, X))^N.$$

Summing over all genus g 2-connected graphs we obtain the following expression of Ψ_g via $\Psi_g^{(1)}$

Proposition 3.1. *For $g > 0$*

$$(3.13) \quad \Psi_g(S, X) = \Psi_g^{(1)}(S, X + S\Phi(S, X)).$$

Consider the set $\mathcal{L}_{N,[ij]}^k$ of colored bipartite connected modular trees Γ consisting of a chain of k genus 0 vertices $V_a(\Gamma)$ interleaving with $k - 1$ two-valent s -vertices $v' \in V_s(\Gamma)$ having two marked ordered tails i and j incident to the farthest vertices of Γ . Put $\mathcal{L}_{N,[ij]} = \bigcup_{k \geq 0} \mathcal{L}_{N,[ij]}^k$ and $\mathcal{L}_{[ij]} = \bigcup_{N \geq 0} \mathcal{L}_{N,[ij]}$. Note that such graphs have no nontrivial automorphisms (at least for the case of ordered tails); reading all the vertices of Γ along the chain starting from the vertex incident to the first marked tail matches $\mu(\Gamma) \frac{X^N}{N!}$ to a certain summand of the ij element of the matrix

$$(3.14) \quad \underbrace{HSHSH \dots SH}_{k \text{ times } H},$$

where $H = H(X)$ is the the Hessian matrix $(\frac{\partial^2 U_0(X)}{\partial x_i \partial x_j})$. Consider the matrix $\Upsilon_k(X)$ of generating series defined by

$$(3.15) \quad \Upsilon_k(X)_{ij} = \sum_{N \geq 0} \sum_{\Gamma \in \mathcal{L}_{N,[ij]}^k} \mu(\Gamma) \frac{X^N}{N!}.$$

and the generating series

$$(3.16) \quad \Upsilon(X)_{ij} = \sum_{\Gamma \in \mathcal{L}_{[ij]}} \mu(\Gamma) \frac{X^{N(\Gamma)}}{N(\Gamma)!}.$$

Summing over all the trees in $\mathcal{L}_{N,[ij]}^k$ we shall get all the summands of the corresponding term of (3.14). Therefore we have obtained the following formula.

Proposition 3.2.

$$(3.17) \quad \Upsilon_k(X) = \underbrace{HSHSH \dots SH}_{k \text{ times } H}.$$

Corollary 3.3.

$$(3.18) \quad \Upsilon(X) = H + HSH + HSHSH + \dots = H(E - SH)^{-1} = (E - HS)^{-1} H.$$

Now it is very easy to give a purely combinatorial proof of the formula (3.9). According to (2.5) the ij component of the matrix $\Theta(S, X)$ is the generating function for the trees from $\bigcup_{N \geq 0} \mathcal{B}_{0,N,[ij]}$. In any tree $\Gamma \in \mathcal{B}_{0,N,[ij]}$ there is a unique chain connecting the two marked edges; this chain we may consider as an element of $\mathcal{L}_{N,[ij]}^k$ (k is the length of this chain). Thus we have defined a mapping

$$(3.19) \quad c_1 : \mathcal{B}_{0,N,[ij]} \rightarrow \mathcal{L}_{N,[ij]}.$$

As in the proof of the proposition 3.1 for any chain $\Lambda \in \mathcal{L}_{N,[ij]}$ all the graphs in $c_1^{-1}(\Lambda)$ are constructed from Λ by clutching arbitrary genus 0 trees with one marked tail to some of the tails of Λ (inserting a two-valent s -vertex between the corresponding tail of Γ and the marked edge of the tree). Therefore

$$(3.20) \quad \sum_{\Gamma \in c_1^{-1}(\Lambda)} \frac{\mu(\Gamma)}{|\text{Aut } \Gamma|} X^{N(\Gamma)} = \frac{\mu(\Lambda)}{|\text{Aut } \Lambda|} (X + S\Phi(S, X))^{N(\Lambda)}$$

and

(3.21)

$$\begin{aligned}\Theta(S, X)_{ij} &= \sum_{\Gamma \in \mathcal{B}_{0, [ij]}} \frac{\mu(\Gamma)}{|\text{Aut } \Gamma|} X^{N(\Gamma)} = \sum_{\Lambda \in \mathcal{L}_{[ij]}} \left(\sum_{\Gamma \in c_1^{-1}(\Lambda)} \frac{\mu(\Gamma)}{|\text{Aut } \Gamma|} X^{N(\Gamma)} \right) = \\ &= \sum_{\Lambda \in \mathcal{L}_{[ij]}} \frac{\mu(\Lambda)}{|\text{Aut } \Lambda|} (X + S\Phi(S, X))^{N(\Lambda)} = \Upsilon(X + S\Phi(S, X))_{ij}.\end{aligned}$$

Using (3.18) and multiplying by S we get

(3.22)

$$\begin{aligned}S\Theta(S, X) &= SH(X + S\Phi(S, X)) + SH(X + S\Phi(S, X))SH(X + S\Phi(S, X)) + \\ &+ SH(X + S\Phi(S, X))SH(X + S\Phi(S, X))SH(X + S\Phi(S, X)) + \dots\end{aligned}$$

and finally adding E we obtain (3.9):

$$(3.23) \quad E + S\Theta(S, X) = (E - SH(X + S\Phi(S, X)))^{-1}.$$

This completes our study of the first term of the genus expansion (the so-called "tree approximation"). Now let us go on with subsequent terms. Our next step is to describe $\Psi_g^{(1)}$. First let us study the case $g = 1$. The set of 2-connected genus 1 graphs splits into two parts $\mathcal{B}_{1,N}^{(1)} = \mathcal{B}'_{1,N} \sqcup \mathcal{B}''_{1,N}$, where $\mathcal{B}'_{1,N}$ is the set of all 2-connected genus 1 graphs having only genus 0 vertices. A connected genus 1 graph may have at most one vertex of positive genus; if such a vertex exists it should have genus 1. So a graph $\Gamma \in \mathcal{B}''_{1,N}$ has no cycles, therefore it has no edges. Hence for each N $\mathcal{B}''_{1,N}$ consists of one graph, namely single genus 1 vertex with $|N|$ tails colored by N . Therefore

$$(3.24) \quad \sum_{N \geq 0} \left(\sum_{\Gamma \in \mathcal{B}''_{1,N}} \frac{\mu(\Gamma)}{|\text{Aut } \Gamma|} \right) X^N = \Psi_1(0, X) = U_1(X).$$

If a genus 1 graph has only genus 0 vertices then it must have exactly one cycle. Therefore a graph $\Gamma \in \mathcal{B}'_{1,N}$ consists of one cycle having $k > 0$ vertices $v'' \in V_a(\Gamma)$ interleaving with k two-valent s -vertices $v' \in V_s(\Gamma)$. Denote the set of such graphs by $\mathcal{B}_{1,N}^{k(1)}$; the set of such graphs with the additional choice of one two-valent vertex $v''_0 \in V_s(\Gamma)$ and of an orientation of the cycle will be denoted by $\overline{\mathcal{B}}_{1,N}^{k(1)}$.

The $2k$ -sheet covering $\overline{\mathcal{B}}_{1,N}^{k(1)} \rightarrow \mathcal{B}_{1,N}^{k(1)}$ is compatible with the automorphisms of the corresponding graphs. Therefore $\sum_{\Gamma \in \mathcal{B}_{1,N}^{k(1)}} \frac{\mu(\Gamma)}{|\text{Aut } \Gamma|} X^N =$

$\frac{1}{2k} \sum_{\bar{\Gamma} \in \mathcal{B}_{1,N}^{k(1)}} \frac{\mu(\Gamma)}{|\text{Aut } \Gamma|} X^N$ where $\bar{\Gamma}$ denotes a graph Γ together with the described additional structure. Deletion of the vertex v_0'' defines a bijection

$$(3.25) \quad \mathcal{B}_{1,N}^{k(1)} \rightarrow \bigcup_{ij} \mathcal{L}_{N,[ij]}^k,$$

therefore

$$(3.26) \quad \sum_{\Gamma \in \mathcal{B}_{1,N}^{k(1)}} \frac{\mu(\Gamma)}{|\text{Aut } \Gamma|} X^N = \frac{1}{2k} \text{tr} (S H(X))^k.$$

Summing for all k we obtain the formula for Ψ_1 (the so-called "one-loop approximation").

Proposition 3.3.

$$(3.27) \quad \Psi_1^{(1)}(S, X) = U_1(X) - \frac{1}{2} \text{tr} \ln(E - S H(X)).$$

Corollary 3.4.

$$(3.28) \quad \Psi_1(S, X) = U_1(X + S\Phi(S, X)) - \frac{1}{2} \text{tr} \ln(E - SH(X + S\Phi(S, X))).$$

A pair of two-valent s -vertices $v'_1, v'_2 \in V_s(\Gamma)$ of a colored bipartite connected and 2-connected modular graph $\Gamma \in \mathcal{B}_{g,N}^{(1)}$ with $g \geq 1$ will be called a *2-cut* if deleting of v'_1 and v'_2 disconnects the graph and at least one of the two new connected components has genus zero. Note that for $g > 1$ at most one of the two components may have genus zero and that the genus 0 component is a tree from $\mathcal{L}_{N',[ij]}^{k'}$ for some N', k' . A graph without 2-cuts will be called *3-connected*; the set of 3-connected genus g graphs having n_i tails of the color i will be denoted by $\mathcal{B}_{g,N}^{(2)}$; $\mathcal{B}_g^{(2)} = \bigcup_{N \geq 0} \mathcal{B}_{g,N}^{(2)}$. The 2-cuts of a given graph $\Gamma \in \mathcal{B}_{g,N}^{(1)}$ are partially ordered by the inclusion relation of the corresponding genus 0 components. Consider all the maximal 2-cuts. Replacing each corresponding maximal genus 0 component by a new two-valent s -vertex we obtain a 3-connected graph $\bar{\Gamma} \in \mathcal{B}_{g,N}^{(2)}$. This provides the mapping

$$(3.29) \quad c_2: \mathcal{B}_g^{(1)} \rightarrow \mathcal{B}_g^{(2)}.$$

Pick a graph $\Gamma \in \mathcal{B}_g^{(2)}$. The preimage $c_2^{-1}(\bar{\Gamma})$ consists of all graphs obtained from Γ by replacing some of the two-valent s -vertices by arbitrary trees from $\mathcal{L}_{N,[ij]}^k$ (bounded by two two-valent s -vertices on the

clutching positions). Therefore

$$(3.30) \quad \left(\sum_{\Gamma \in c_2^{-1}(\Gamma)} \frac{\mu(\Gamma)}{|\text{Aut } \Gamma|} \right) X^{N(\Gamma)}$$

is obtained from $\frac{\mu(\Gamma)}{|\text{Aut } \Gamma|} X^N$ by substituting

$$(3.31) \quad (S + SHS + SHSHS + \dots)_{ij}$$

instead of all s_{ij} . Note that the matrix in (3.31) may be expressed as

$$(3.32) \quad S + SHS + SHSHS + \dots = S(E - HS)^{-1} = (E - SH)^{-1} S.$$

This enables to express the function Ψ in terms of the generating function for 3-connected graphs

$$(3.33) \quad \Psi_g^{(2)}(S, X) = \sum_{N \geq 0} \left(\sum_{\Gamma \in \mathcal{B}_{g,N}^{(2)}} \frac{\mu(\Gamma)}{|\text{Aut } \Gamma|} \right) X^N.$$

Proposition 3.4. *For $g > 1$*

$$(3.34) \quad \Psi_g^{(1)}(S, X) = \Psi_g^{(2)}((E - SH(X))^{-1} S, X).$$

Corollary 3.5. *For $g > 1$*

$$(3.35) \quad \Psi_g(S, X) = \Psi_g^{(2)}((E - SH(X + S\Phi(S, X)))^{-1} S, X + S\Phi(S, X)).$$

Now we are left to describe $\Psi_g^{(2)}$. Deletion of all the tails defines the mapping

$$(3.36) \quad c_3: \mathcal{B}_g^{(2)} \rightarrow \mathcal{B}_{g,0}^{(2)},$$

where $\mathcal{B}_{g,0}^{(2)}$ is the set of genus $g > 1$ 3-connected graphs without tails. Note that a graph is 3-connected if and only if it is stable. We shall denote the set of stable closed graphs $\mathcal{B}_{g,0}^{(2)}$ by \mathcal{A}_g . Pick a stable closed graph $\Gamma \in \mathcal{A}_g$ and a vertex $v \in V_a(\Gamma)$; let $N(v)$ ($|N(v)| \geq 3$ for $g = 0$) be the multiindex of its valences. Note that $a_{g(v), N(v)} = \frac{\partial^{|N(v)|} U_{g(v)}}{\partial X^{N(v)}}(0)$, and the same vertex v in different graphs from $c_3^{-1}(\Gamma)$ corresponds to certain terms of the expansion of $\frac{\partial^{|N(v)|} U_{g(v)}(X)}{\partial X^{N(v)}}$. Thus it is not hard to verify that

$$(3.37) \quad \left(\sum_{\Delta \in c_3^{-1}(\Gamma)} \frac{\mu(\Delta)}{|\text{Aut } \Delta|} \right) X^{N(\Delta)} = \frac{1}{|\text{Aut } \Gamma|} \prod_{v \in V_a(\Gamma)} \frac{\partial^{|N(v)|} U_{g(v)}}{\partial X^{N(v)}}(X).$$

The product on the right side of (3.37) looks like the second product in the definition of $\mu(\Gamma)$ (1.5) with $\frac{\partial^{|N|} U_g(X)}{\partial X^N}$ substituted instead of the variables $a_{g,N}$. Therefore, defining for $g \geq 1$ the generating functions P_g for stable closed graphs by

$$(3.38) \quad P_g(\{a_{m,N}\}, S) = \sum_{\Gamma \in \mathcal{A}_g} \frac{\mu(\Gamma)}{|\text{Aut } \Gamma|} = \Psi_g^{(2)}(\{a_{m,N}\}, S, 0)$$

we obtain the following expression for $\Psi_g^{(2)}$:

$$(3.39) \quad \Psi_g^{(2)}(\{a_{m,N}\}, S, X) = \sum_{N \geq 0} \left(\sum_{\Gamma \in \mathcal{B}_{g,N}^{(2)}} \frac{\mu(\Gamma)}{|\text{Aut } \Gamma|} \right) X^N = \\ = P_g \left(\left\{ a_{m,N} := \frac{\partial^{|N|} U_m(X)}{\partial X^N} \right\}, S \right).$$

Thus we are able to express the generating functions Ψ_g in terms of the generating functions for stable closed graphs P_g .

Theorem 3.1. *For $g > 1$*

$$(3.40) \quad \Psi_g(S, X) = \\ = P_g \left(\left\{ a_{g,N} := \frac{\partial^{|N|} U_g(X + S\Phi(S, X))}{\partial X^N} \right\}, (E - SH(X + S\Phi(S, X)))^{-1} S \right).$$

Note that for each $g > 1$ the set of stable closed graphs \mathcal{A}_g is finite: for $g > 1$ a stable closed genus g graph has at most $3g - 3$ two-valent s -vertices. Hence $P_g(\{a_{m,N}\}, S)$ is a polynomial in s_{ij} and $\{a_{m,N}\}$ for $|N| \leq 2g - 2$ ($|N| \leq 2$ for $g = 1$) and $0 \leq m \leq g$. It has degree $3g - 3$ as a polynomial in s_{ij} ; for combinatorial case the degree of all terms is at least g . We shall call the polynomials $P_g(\{a_{m,N}\}, S)$ *stable graph polynomials*. For instance the first stable graph polynomial for $g = 2$

$$(3.41) \quad P_2 = a_{2,0} + \sum_{i,j} \frac{1}{\{ij\}!} a_{1,\{i\}} a_{1,\{j\}} s_{ij} + \sum_{i,j} \frac{1}{\{ij\}!} a_{1,\{ij\}} s_{ij} + \\ + \sum_{i,j,k,l} \frac{1}{\{kl\}!} a_{1,\{i\}} a_{0,\{jkl\}} s_{ij} s_{kl} + \sum_{i,j,k,l,p,q} \frac{1}{\text{Aut } \Gamma_5} a_{0,\{ijk\}} a_{0,\{lpq\}} s_{ij} s_{kl} s_{pq} + \\ + \sum_{i,j,k,l,p,q} \frac{1}{\text{Aut } \Gamma_6} a_{0,\{ijk\}} a_{0,\{lpq\}} s_{ip} s_{kl} s_{jq} + \sum_{i,j,k,l} \frac{1}{\text{Aut } \Gamma_7} a_{0,\{ijkl\}} s_{ij} s_{kl}.$$

has seven groups of terms corresponding to the seven possible genus 2 graphs; different terms in each group correspond to different ways of

coloring edges of the given graph. All the coefficients are the inverses to the number of automorphisms of the corresponding graph; in the last three terms we do not indicate an explicit expression of dependence of these numbers on the way of coloring. For combinatorial case P_2 has only three last terms.

It is not hard to verify that the stable graph polynomials are homogeneous in the following sense.

Proposition 3.5. *Define the grading of the polynomial ring $\mathbb{C}[\{s_{ij}\}, \{a_{m,N}\}]$ by*

$$(3.42) \quad \deg a_{g,N} = 1 - |N| - g \quad \text{and} \quad \deg s_{ij} = 1.$$

Then stable graph polynomial $P_g(\{a_{g,N}\}, S)$ is homogeneous polynomial of degree $1 - g$.

4. STABLE GRAPH POLYNOMIALS

Next let us derive the recurrence for the stable graph polynomials. The idea of it is quite similar to the proof of theorem 1.2: we delete one s -vertex of a given stable closed genus g graph and obtain a genus $g - 1$ graph with two tails. Unlike the cases considered in the theorems 1.2 the new graph does not correspond to the same generating function for genus $g - 1$, because it is not closed and may be not connected. First let us study the latter case. Let Γ be a stable closed genus $g > 1$ graph and assume that deleting of some s -vertex disconnects it into the disjoint union of two connected graphs Γ' and Γ'' . Then both Γ' and Γ'' have positive genus and each of the two has exactly one tail. Let us denote by $\mathcal{C}_{g,\{i\}}$ the set of all genus $g \geq 1$ graphs with the only tail of color i obtained from stable closed graphs in the described way. Pick a graph $\Gamma \in \mathcal{C}_{g,\{i\}}$. For $g > 1$ deletion of its only tail provides a stable closed graph unless the vertex v_0 incident to the tail was a trivalent genus 0 vertex. But in the latter case we obtain a stable closed graph by substituting an s -vertex instead of the subgraph consisting of the vertex v_0 together with the two s -vertices adjacent to v_0 (see Fig. 2). Thus for $g > 1$ we have defined a mapping

$$(4.1) \quad c_5: \mathcal{C}_{g,\{i\}} \rightarrow \mathcal{B}_{g,0}^{(2)}.$$

The generating function $(\mu(\Gamma))$ is defined in (1.5))

$$(4.2) \quad Q_g^{(i)}(\{a_{m,N}\}, S) = \sum_{\Gamma \in \mathcal{C}_{g,\{i\}}} \frac{\mu(\Gamma)}{|\text{Aut } \Gamma|}$$

is a derivative of P_g in the following sense. For $1 \leq k \leq r$ define the differentiation D_k of the ring of polynomials in all $a_{m,N}$ and s_{ij} by its

action on its generators:

$$(4.3) \quad D_k(a_{m,N}) = a_{m,N+\{k\}}$$

$$(4.4) \quad D_k(s_{ij}) = \sum_{p,q} s_{ip}s_{jq}a_{0,\{pqk\}}$$

Proposition 4.1. *The differentiation D_k is homogeneous and has degree -1 .*

Using the mapping (4.1) it is not hard to verify the following statement.

Proposition 4.2. *For $g > 1$*

$$(4.5) \quad Q_g^{(i)}(\{a_{m,N}\}, S) = D_i(P_g(\{a_{m,N}\}, S))$$

For $g = 1$ the mapping (4.1) is not well-defined but it is not hard to list all the graphs of $\mathcal{C}_{1,\{i\}}$ explicitly. In fact there are only two such graphs: single genus 1 vertex with one color i tail and a length one cycle with one s -vertex and one genus 0 trivalent vertex having one tail of the color i . Therefore

$$(4.6) \quad Q_1^{(i)}(\{a_{m,N}\}, S) = a_{1,\{i\}} + \sum_{p,q} \frac{1}{\{pq\}!} s_{p,q} a_{0,\{ipq\}},$$

which may be considered as a formal definition of $D_i(P_1)$ (whereas P_1 does not exist).

Next let us consider the second possibility. Pick a stable closed genus $g > 2$ graph and assume that deleting of some s -vertex does not disconnect it. Let us denote by $\mathcal{C}_{g,\{ij\}}$ the set of all genus $g > 1$ graphs having exactly two tails of colors i and j obtained from stable closed graphs in the described way. Our next purpose is to define for $g > 1$ a mapping

$$(4.7) \quad c_4: \mathcal{C}_{g,\{ij\}} \rightarrow \mathcal{B}_{g,0}^{(2)}.$$

First let us assume that the two tails are not attached to the same trivalent genus 0 vertex. In this case the mapping (4.7) may be described as the result of twice repeated operations used in the definition of the mapping c_5 (4.1). This corresponds to double differentiation $\frac{1}{\{ij\}!} D_i D_j$. Next consider a graph $\Gamma \in \mathcal{C}_{g,\{ij\}}$ having two tails attached to the same trivalent genus 0 vertex v_0 . Then 0 is adjacent to exactly one s -vertex which connects it to the remaining part of the graph. Removal of this s -vertex (together with v_0) provides a graph $\bar{\Gamma} \in \mathcal{C}_{g,\{m\}}$ for some m .

This enables to describe the generating function

$$(4.8) \quad R_g^{(ij)}(\{a_{m,N}\}, S) = \sum_{\Gamma \in \mathcal{C}_{g,\{ij\}}} \frac{\mu(\Gamma)}{|\text{Aut } \Gamma|}$$

using the mapping c_4 (4.7).

Proposition 4.3. *For $g > 1$*

$$(4.9) \quad R_g^{(ij)}(\{a_{m,N}\}, S) = \frac{1}{\{ij\}!} \left[D_i D_i (P_g(\{a_{m,N}\}, S) + \sum_{p,q} D_p (P_g(\{a_{m,N}\}, S)) s_{pq} a_{0,\{ijq\}} \right].$$

For $g = 1$ the mapping (4.7) is not well-defined but it is not hard to list all the graphs of $\mathcal{C}_{1,\{ij\}}$ explicitly. In fact there are only three such graphs corresponding to the three terms of the following expression

$$(4.10) \quad \begin{aligned} R_1^{(ij)}(\{a_{m,N}\}, S) &= \frac{1}{\{ij\}!} \left[\sum_{p,q,u,t} \frac{1}{\{\{pq\}\{ut\}\}!} s_{pu} s_{qt} a_{0,\{ipq\}} a_{0,\{jut\}} + \right. \\ &\quad \left. + \sum_{p,q} \left(a_{1,\{p\}} s_{pq} a_{0,\{ijq\}} + \sum_{u,t} \frac{1}{\{ut\}!} s_{ut} a_{0,\{utp\}} s_{pq} a_{0,\{ijq\}} \right) \right] = \\ &= \frac{1}{\{ij\}!} \left[\sum_{p,q,u,t} \frac{1}{\{\{pq\}\{ut\}\}!} s_{pu} s_{qt} a_{0,\{ipq\}} a_{0,\{jut\}} + \sum_{p,q} D_p (P_1) s_{pq} a_{0,\{ijq\}} \right], \end{aligned}$$

where $\{\{pq\}\{ut\}\}!$ means 2 for $p = q$ and $u = t$ and 1 otherwise. Note that in the last expression the term $\sum_{p,q,u,t} \frac{1}{\{\{pq\}\{ut\}\}!} s_{pu} s_{qt} a_{0,\{ipq\}} a_{0,\{jut\}}$ may be considered as a formal definition of $D_i D_j (P_1)$ (different from $D_i (D_j (P_1))$ and $D_j (D_i (P_1))$ which are not equal).

Now we are prepared to present the recurrence for stable graph polynomials. Deletion of an s -vertex from a given stable closed genus g graph provides either a connected graph from $\mathcal{C}_{g-1,\{ij\}}$ or a pair of connected graph from $\mathcal{C}_{m,\{i\}}$ and $\mathcal{C}_{g-m,\{j\}}$ for some $1 \leq m \leq g-1$. Therefore

$$(4.11) \quad \frac{\partial P_g}{\partial s_{ij}} = \frac{1}{\{ij\}!} \left[R_{g-1}^{(ij)}(\{a_{m,N}\}, S) + \sum_{m=1}^{g-1} Q_m^{(i)}(\{a_{m,N}\}, S) Q_{g-m}^{(j)}(\{a_{m,N}\}, S) \right].$$

Substituting (4.5), (4.6) and (4.9) we get the desired recurrence.

Theorem 4.1. *For $g > 2$ stable graph polynomials $P_g(\{a_{m,N}\}, S)$ for colored bipartite graphs are given by the recurrences (for each pair ij)*

$$(4.12) \quad \frac{\partial P_g}{\partial s_{ij}} = \frac{1}{\{ij\}!} \left[D_i D_j (P_{g-1}) + \sum_{p,q} D_p (P_{g-1}) s_{pq} a_{0,\{ijq\}} + \right. \\ \left. + D_i (P_{g-1}) \left(a_{1,\{j\}} + \sum_{p,q} \frac{1}{\{pq\}!} s_{p,q} a_{0,\{j pq\}} \right) + \right. \\ \left. + D_j (P_{g-1}) \left(a_{1,\{i\}} + \sum_{p,q} \frac{1}{\{pq\}!} s_{p,q} a_{0,\{i pq\}} \right) + \sum_{m=2}^{g-2} D_i (P_m) D_j (P_{g-m}) \right].$$

and by the initial condition $P_g(\{a_{m,N}\}, 0) = a_{g,0}$, where the differentiations D_i are given by (4.3) and (4.4) and P_2 is defined in (3.41).

Regardless of the absence of genus 1 stable closed graphs we may start the recurrence (4.12) from $g = 1$ formally putting

$$(4.13) \quad D_i (P_1) = a_{1,\{i\}} + \sum_{p,q} \frac{1}{\{pq\}!} s_{p,q} a_{0,\{i pq\}}.$$

Then for any $g > 1$

$$(4.14) \quad \frac{\partial P_g}{\partial s_{ij}} = \frac{1}{\{ij\}!} \left[D_i D_j (P_{g-1}) + \sum_{p,q} D_p (P_{g-1}) s_{pq} a_{0,\{ijq\}} + \right. \\ \left. + \sum_{m=1}^{g-1} D_i (P_m) D_j (P_{g-m}) \right].$$

5. COUNTING FUNCTIONS FOR COMBINATORIAL GRAPHS.

For $r = 1$ we have one variable s and two-index variables $a_{g,n}$. The differentiation D of the ring of polynomials in s and $a_{g,n}$ is defined by

$$(5.1) \quad D(a_{g,n}) = a_{g,n+1}$$

$$(5.2) \quad D(s) = s^2 a_{0,3}$$

The polynomial P_2 is given by (3.41):

$$(5.3) \quad P_2 = a_{2,0} + \frac{1}{2} a_{1,1}^2 s + \frac{1}{2} a_{1,2} s + \frac{1}{2} a_{1,1} a_{0,3} s^2 + \frac{1}{8} a_{0,4} s^2 + \left(\frac{1}{12} + \frac{1}{8} \right) a_{0,3}^2 s^3,$$

and for combinatorial case ($a_{g,n} = 0$ for $g > 0$)

$$(5.4) \quad P_2 = \frac{1}{8}a_{0,4}s^2 + \frac{5}{24}a_{0,3}^2s^3.$$

The recurrence (4.12) becomes

$$(5.5) \quad \frac{\partial P_g}{\partial s} = \frac{1}{2} \left[D^2(P_{g-1}) + 2D(P_{g-1})(a_{1,1} + sa_{0,3}) + \sum_{m=2}^{g-2} D(P_m)D(P_{g-m}) \right]$$

and the formula (3.40) looks like

$$(5.6) \quad \Psi_g(s, x) = P_g \left(\left\{ a_{g,n} := \frac{d^n U_g}{dx^n}(x + s\Phi(s, x)) \right\}, \frac{s}{1 - sH(x + s\Phi(s, x))} \right).$$

For the counting functions for combinatorial graphs we put

$$(5.7) \quad a_{g,n}^{\text{comb}} = \begin{cases} 1 & \text{if } g = 0 \\ 0 & \text{otherwise} \end{cases}$$

which we may write uniformly as $a_{g,n}^{\text{comb}} = \delta_{g0}$. Then the initial conditions (see (??) — (1.23)) are

$$(5.8) \quad \begin{aligned} U_g^{\text{comb}}(x) &= 0 & \text{for } g > 0, \\ U_0^{\text{comb}}(x) &= e^x, \end{aligned}$$

and hence $F^{\text{comb}}(x) = H^{\text{comb}}(x) = e^x$. The counting series for all trees satisfies the functional equation (3.1):

$$(5.9) \quad \Phi^{\text{comb}}(s, x) = e^{x+s\Phi^{\text{comb}}(s,x)}$$

and therefore

$$(5.10) \quad \frac{d^n U_g^{\text{comb}}}{dx^n}(x + s\Phi^{\text{comb}}(s, x)) = 0 \quad \text{for } g > 0,$$

and

$$(5.11) \quad \frac{d^n U_0^{\text{comb}}}{dx^n}(x + s\Phi^{\text{comb}}(s, x)) = e^{x+s\Phi^{\text{comb}}(s,x)} = \Phi^{\text{comb}}(s, x) \quad \text{for } g = 0,$$

which we may write in a uniform way as $a_{g,n} := \Phi^{\text{comb}}(s, x)\delta_{g0}$. Denote the second argument of (5.6) by Y :

$$(5.12) \quad Y = \frac{s}{1 - sH(x + s\Phi(s, x))} := \frac{s}{1 - se^{x+s\Phi^{\text{comb}}(s,x)}} = \frac{s}{1 - s\Phi^{\text{comb}}(s, x)}.$$

Therefore for the counting function for combinatorial graphs

$$(5.13) \quad \begin{aligned} \Psi_g^{\text{comb}}(s, x) &= \Psi_g(\{a_{g,n} := \delta_{g0}\}, s, x) = \\ &= P_g\left(\{a_{g,n} := \Phi^{\text{comb}}(s, x)\delta_{g0}\}, \frac{s}{1 - s\Phi^{\text{comb}}(s, x)}\right). \end{aligned}$$

Recall that stable graph polynomials are homogenous and have degree $1 - g$ (see proposition 3.5) with respect to the grading (3.42). By definition of stable graph polynomials (3.38):

$$(5.14) \quad P_g(\{a_{m,N}\}, s) = \sum_{\Gamma \in \mathcal{A}_g} \frac{\mu(\Gamma)}{|\text{Aut } \Gamma|},$$

where \mathcal{A}_g^k is the set of genus g stable closed graphs. Denote the set of genus g stable closed graphs with k edges by \mathcal{A}_g^k . A graph $\Gamma \in \mathcal{A}_g^k$ has $k - g + 1$ vertices (combinatorial graphs have no vertices of higher genus), so $\mu(\Gamma) = s^k \prod_{i=1}^{k-g+1} a_{0,n_i}$, where n_i are the valences of the vertices. Therefore the counting function for combinatorial graphs

$$(5.15) \quad \begin{aligned} \Psi_g^{\text{comb}}(s, x) &= P_g(\{a_{m,N} := \Phi^{\text{comb}}(s, x)\delta_{m0}\}, Y) = \\ &= \sum_k Y^k \left(\sum_{\Gamma \in \mathcal{A}_g^k} \frac{1}{|\text{Aut } \Gamma|} \prod_{i=1}^{k-g+1} a_{0,n_i} \right) = \sum_k Y^k \left(\sum_{\Gamma \in \mathcal{A}_g^k} \frac{1}{|\text{Aut } \Gamma|} \Phi^{\text{comb}}(s, x)^{k-g+1} \right) = \\ &= \frac{1}{\Phi(s, x)^{g-1}} \sum_k (\Phi^{\text{comb}}(s, x)Y)^k \left(\sum_{\Gamma \in \mathcal{A}_g^k} \frac{1}{|\text{Aut } \Gamma|} \right) = \\ &= \frac{1}{\Phi^{\text{comb}}(s, x)^{g-1}} P_g\left(\{a_{m,N} := \delta_{m0}\}, \frac{s\Phi^{\text{comb}}(s, x)}{1 - s\Phi^{\text{comb}}(s, x)}\right). \end{aligned}$$

Note that the polynomials

$$(5.16) \quad P_g^{\text{comb}}(s) = P_g(\{a_{m,N} := \delta_{m0}\}, s)$$

are the generating functions for combinatorial stable closed graphs. Thus we have proved the second part of the theorem 1.6.

Next let us prove the first part of this theorem. The only problem in deriving an explicit recurrence for $P_g^{\text{comb}}(s)$ from (4.12) is how to express the result of substitution $a_{m,N} := \delta_{m0}$ into $D(P_g)$ and $D^2(P_g)$ in terms of $P_g^{\text{comb}}(s)$.

Consider the polynomial ring $\mathbb{C}[\alpha, Y]$ with the grading

$$(5.17) \quad \deg Y = 1 \quad \text{and} \quad \deg \alpha = -1$$

and differentiation δ^{comb} of this ring defined by its action on the generators

$$(5.18) \quad \delta_{\text{comb}}(\alpha) = \alpha \quad \text{and} \quad \delta_{\text{comb}}(Y) = Y^2\alpha.$$

Then δ_{comb} is homogeneous of degree 0. Define a ring homomorphism

$$(5.19) \quad f^{\text{comb}}: \mathbb{C}[\{a_{g,n}\}, s_{ij}] \rightarrow \mathbb{C}[\alpha, Y]$$

by its action on the generators:

$$(5.20) \quad \begin{aligned} f^{\text{comb}}(a_{g,n}) &= 0 \quad \text{for } g > 0, \\ f^{\text{comb}}(a_{0,n}) &= \alpha \\ f^{\text{comb}}(s_{ij}) &= Y. \end{aligned}$$

Evidently $D \circ f^{\text{comb}} = f^{\text{comb}} \circ \delta_{\text{comb}}$, therefore $D^2 \circ f^{\text{comb}} = f^{\text{comb}} \circ \delta_{\text{comb}}^2$. The calculation (5.15) shows that $f^{\text{comb}}(P_g)$ is homogeneous polynomial of degree $g - 1$, hence the same is true about $f^{\text{comb}}(D(P_g)) = \delta_{\text{comb}}(f^{\text{comb}}(P_g))$. But for any degree $g - 1$ homogeneous polynomial $W \in \mathbb{C}[\alpha, Y]$

$$(5.21) \quad \begin{aligned} \delta_{\text{comb}}(W) &= Y^2\alpha \frac{\partial W}{\partial Y} + \alpha \frac{\partial W}{\partial \alpha} = \\ &= Y^2\alpha \frac{\partial W}{\partial Y} + Y \frac{\partial W}{\partial Y} - Y \frac{\partial W}{\partial Y} + \alpha \frac{\partial W}{\partial \alpha} = \\ &= Y(Y\alpha + 1) \frac{\partial W}{\partial Y} - (g - 1)W. \end{aligned}$$

(On the last step we use the Euler formula for homogeneous polynomials: $Y \frac{\partial W}{\partial Y} - \alpha \frac{\partial W}{\partial \alpha} = (g - 1)W$.) Now we can apply (5.21) to the combinatorial stable graph polynomials

$$(5.22) \quad P_g^{\text{comb}}(s) = (f^{\text{comb}} P_g)(1, s)$$

and get the following recurrence for P_g^{comb} .

Proposition 5.1. *The combinatorial stable graph polynomials P_g^{comb} (see (5.16)) satisfy the following recurrence*

$$(5.23) \quad \frac{dP_g^{\text{comb}}}{ds} = \frac{1}{2} \left[D_{\text{comb}}^2(P_{g-1}^{\text{comb}}) + 2s D_{\text{comb}}(P_{g-1}^{\text{comb}}) + \sum_{m=2}^{g-2} D_{\text{comb}}(P_m^{\text{comb}}) D_{\text{comb}}(P_{g-m}^{\text{comb}}) \right],$$

where

$$(5.24) \quad D_{\text{comb}} = s(s+1) \frac{d}{ds} - (g-1).$$

Here we present the explicit form of the recurrence 5.23 and the polynomials P_g for $g \leq 6$:

$$(5.25) \quad \frac{d}{ds} P_g^{\text{comb}}(s) = \frac{1}{2} \left[s^2(s+1)^2 \frac{d^2}{ds^2} P_{g-1}^{\text{comb}}(s) - \right. \\ \left. - s(s+1)(2g-4s-5) \frac{d}{ds} P_{g-1}^{\text{comb}}(s) + (g-1) P_{g-1}^{\text{comb}}(s) + \right. \\ \left. + \sum_{m=2}^{g-2} \left(\left(s(s+1) \frac{d}{ds} P_m^{\text{comb}}(s) - (m-1) P_m^{\text{comb}}(s) \right) \times \right. \right. \\ \left. \left. \times \left(s(s+1) \frac{d}{ds} P_{g-m}^{\text{comb}}(s) - (g-m-1) P_{g-m}^{\text{comb}}(s) \right) D(P_m) D(P_{g-m}) \right) \right]$$

Here we present the polynomials P_g^{comb} for $g \leq 6$:

$$P_2^{\text{comb}} = \frac{5}{24} s^3 + \frac{1}{8} s^2 \\ P_3^{\text{comb}} = \frac{5}{16} s^6 + \frac{25}{48} s^5 + \frac{11}{48} s^4 + \frac{1}{48} s^3 \\ P_4^{\text{comb}} = \frac{1105}{1152} s^9 + \frac{985}{384} s^8 + \frac{1373}{576} s^7 + \frac{515}{576} s^6 + \frac{223}{1920} s^5 + \frac{1}{384} s^4 \\ P_5^{\text{comb}} = \frac{565}{128} s^{12} + \frac{12455}{768} s^{11} + \frac{26581}{1152} s^{10} + \frac{12227}{768} s^9 + \frac{2089}{384} s^8 + \frac{9583}{11520} s^7 + \frac{27}{640} s^6 + \frac{1}{3840} s^5 \\ P_6^{\text{comb}} = \frac{82825}{3072} s^{15} + \frac{387005}{3072} s^{14} + \frac{371195}{1536} s^{13} + \frac{10154003}{41472} s^{12} + \frac{121207}{864} s^{11} + \\ + \frac{519883}{11520} s^{10} + \frac{1573507}{207360} s^9 + \frac{2597}{4608} s^8 + \frac{803}{64512} s^7 + \frac{1}{46080} s^6$$

Similar formulas describe counting functions for combinatorial stable graphs. For this case

$$(5.26) \quad a_{g,n}^{\text{st}} = \begin{cases} 1 & \text{if } g = 0 \text{ and } n \geq 3 \\ 0 & \text{otherwise,} \end{cases}$$

so the initial conditions are

$$(5.27) \quad \begin{aligned} U_g^{\text{st}}(x) &= 0 \quad \text{for } g > 0, \\ U_0^{\text{st}}(x) &= e^x - 1 - x - \frac{x^2}{2}, \end{aligned}$$

and hence $F^{\text{st}}(x) = e^x - 1 - x$ and $H^{\text{st}}(x) = e^x - 1$. The counting series for all stable trees satisfies the functional equation (3.1):

$$(5.28) \quad \Phi^{\text{st}}(s, x) = e^{x+s\Phi^{\text{st}}(s,x)} - 1 - (x + s\Phi^{\text{st}}(s, x))$$

and therefore

$$(5.29) \quad \frac{d^n U_g^{\text{st}}}{dx^n}(x + s\Phi^{\text{st}}(s, x)) = 0 \quad \text{for } g > 0,$$

and

(5.30)

$$\frac{d^n U_0^{\text{st}}}{dx^n} (x + s\Phi^{\text{st}}(s, x)) = e^{x+s\Phi^{\text{st}}(s, x)} = 1+x+(s+1)\Phi^{\text{st}}(s, x) \quad \text{for } g=0, n \geq 3.$$

Since the terms $\frac{d^n U_0^{\text{st}}}{dx^n} (x + s\Phi^{\text{st}}(s, x))$ for $n < 3$ are not involved in the stable graph polynomials for the use in formula (5.6) we may write in a uniform way $a_{g,n} := (1 + x + (s + 1)\Phi^{\text{st}}(s, x))\delta_{g0}$.

The second argument of (5.6) for this case is :

(5.31)

$$\frac{s}{1 - sH^{\text{st}}(x + s\Phi^{\text{st}}(s, x))} = \frac{s}{1 - s(e^{x+s\Phi^{\text{st}}(s, x)} - 1)} = \frac{s}{1 - s(x + (s + 1)\Phi^{\text{st}}(s, x))}.$$

Thus using the same argument as in the proof of (5.15) we get the formula for counting function for combinatorial stable graphs:

$$(5.32) \quad \Psi_g^{\text{st}}(s, x) =$$

$$\begin{aligned} &= P_g \left(\{a_{m,N} := (1 + x + (s + 1)\Phi^{\text{st}}(s, x)) \delta_{m0}\}, \frac{s}{1 - s(x + (s + 1)\Phi^{\text{st}}(s, x))} \right) = \\ &= \frac{1}{(1 + x + (s + 1)\Phi^{\text{st}}(s, x))^{g-1}} P_g \left(\{a_{m,N} := \delta_{m0}\}, \frac{s(1 + x + (s + 1)\Phi^{\text{st}}(s, x))}{1 - s(x + (s + 1)\Phi^{\text{st}}(s, x))} \right) = \\ &= \frac{1}{(1 + x + (s + 1)\Phi^{\text{st}}(s, x))^{g-1}} P_g^{\text{comb}} \left(\frac{s(1 + x + (s + 1)\Phi^{\text{st}}(s, x))}{1 - s(x + (s + 1)\Phi^{\text{st}}(s, x))} \right). \end{aligned}$$

The third part of the theorem 1.6 is proved.

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